

# RATE OF DECAY OF $s$ -NUMBERS

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ABSTRACT. For an operator  $T \in B(X, Y)$ , we denote by  $a_m(T)$ ,  $c_m(T)$ ,  $d_m(T)$ , and  $t_m(T)$  its approximation, Gelfand, Kolmogorov, and absolute numbers. We show that, for any infinite dimensional Banach spaces  $X$  and  $Y$ , and any sequence  $\alpha_m \searrow 0$ , there exists  $T \in B(X, Y)$  for which the inequality

$$3\alpha_{\lceil m/6 \rceil} \geq a_m(T) \geq \max\{c_m(t), d_m(T)\} \geq \min\{c_m(t), d_m(T)\} \geq t_m(T) \geq \alpha_m/9$$

holds for every  $m \in \mathbb{N}$ . Similar results are obtained for other  $s$ -scales.

## 1. INTRODUCTION AND MAIN RESULTS

In this paper, we investigate the existence of an operator  $T \in B(X, Y)$  ( $X$  and  $Y$  are infinite dimensional Banach spaces) whose sequence of  $s$ -numbers  $(s_n(T))$  “behaves like” a prescribed sequence  $(\alpha_n)$ .

For a linear operator  $T$  between Banach spaces  $X$  and  $Y$ , define its *approximation numbers*  $a_n$ , *Kolmogorov numbers*  $d_n$ , *Gelfand numbers*  $c_n$ , *symmetrized* (or *absolute*) *numbers*  $t_n$ , *Weyl numbers*  $x_n$ , *Chang numbers*  $y_n$ , and *Hilbert numbers*  $h_n$ :

$$\begin{aligned}
 (1.1) \quad a_n(T) &= \inf\{\|T - S\| : S \in B(X, Y), \text{rank } S < n\}, \\
 d_n(T) &= \inf\{\|qT\| : q : Y \rightarrow Y/F \text{ quotient map, } \dim F < n\} \\
 &= \inf\{a_n(Tq) : q : \tilde{X} \rightarrow X \text{ quotient map}\}, \\
 c_n(T) &= \inf\{\|T|_E\| : E \hookrightarrow X, \text{codim } E < n\} \\
 &= \inf\{a_n(jT) : j : Y \rightarrow \tilde{Y} \text{ isometry}\}, \\
 t_n(T) &= \inf\{a_n(jTq) : q : \tilde{X} \rightarrow X \text{ quotient map, } j : Y \rightarrow \tilde{Y} \text{ isometry}\}, \\
 x_n(T) &= \inf\{a_n(Tu) : u : \ell_2 \rightarrow X, \|u\| \leq 1\}, \\
 y_n(T) &= \inf\{a_n(vT) : v : Y \rightarrow \ell_2, \|v\| \leq 1\}, \\
 h_n(T) &= \inf\{a_n(vTu) : u : \ell_2 \rightarrow X, v : Y \rightarrow \ell_2, \|u\|\|v\| \leq 1\}.
 \end{aligned}$$

We refer the reader to [3, 24] for general information about these and other  $s$ -numbers. Note that  $t_n(T) \leq \min\{c_n(T), d_n(T)\} \leq \max\{c_n(T), d_n(T)\} \leq a_n(T)$  for any operator  $T$ . We say that an operator  $T$  is *approximable* if  $\lim_n a_n(T) = 0$ . It is well known that  $T$  is compact if and only if  $\lim_n d_n(T) = 0$  if and only if  $\lim_n c_n(T) = 0$ . Any approximable operator is compact, but the converse is not true, due to the existence of Banach spaces failing the Approximation Property.

Throughout the paper, the notation  $\alpha_i \searrow 0$  means that the sequence  $(\alpha_i)$  satisfies  $\alpha_1 \geq \alpha_2 \geq \dots \geq 0$ , and  $\lim_i \alpha_i = 0$ .

We are motivated by Bernstein’s Lethargy Theorem, stating that, for any Banach space  $X$ , any strictly increasing chain of finite dimensional subspaces  $X_1 \hookrightarrow X_2 \hookrightarrow \dots \hookrightarrow X$ , and any sequence  $\alpha_i \searrow 0$ , there exists  $x \in X$  such that  $d(x, X_i) = \alpha_i$  for every  $i$  (for the

proof, see e.g. [25, Section II.5.3]). This theorem was later generalized to the more general class of  $FS$ -spaces [17]. Certain partial results for chains  $X_1 \hookrightarrow X_2 \hookrightarrow \dots \hookrightarrow X$  of infinite dimensional subspaces of a Banach space  $X$  can be found in [25, Section I.6.3]. Related results were obtained for general approximation schemes in [2].

In a similar vein, one can study the existence of operators whose sequences of  $s$ -numbers behave in a prescribed fashion. First results of this kind were obtained in [8]. Among other things, it was proved that, of every pair of infinite dimensional Banach spaces  $(X, Y)$ , and any  $\varepsilon > 0$ , there exist infinite dimensional  $X_0 \hookrightarrow X$  and  $Y_0 \hookrightarrow Y$ , such that for any sequence  $\alpha_i \searrow 0$  there exists  $T \in B(X_0, Y_0)$  with the property that  $\alpha_i \leq a_i(T) \leq (1 + \varepsilon)\alpha_i$  for every  $i$ . Furthermore, for many pairs  $(X, Y)$ , the existence of  $T \in B(X, Y)$  satisfying  $\alpha_i \leq a_i(T) \leq M\alpha_i$  ( $M$  is a constant, depending on  $(X, Y)$ ) is demonstrated. These results were sharpened in [1], where it was shown that, for a certain class of pairs  $(X, Y)$ , for any  $\alpha_i \searrow 0$  there exists  $T \in B(X, Y)$  such that  $a_i(T) = \alpha_i$  for every  $i$ . One should also mention [10], where operators with prescribed eigenvalue sequences are constructed.

The main result of this paper is:

**Theorem 1.1.** *Suppose  $X$  and  $Y$  are infinite dimensional Banach spaces, and  $\alpha_k \searrow 0$ . Then there exists an approximable  $T : X \rightarrow Y$  such that  $\|T\| \leq 2\alpha_1$ , and, for every  $m$ ,  $3\alpha_{\lceil m/6 \rceil} \geq a_m(T) \geq t_m(T) \geq \alpha_m/9$ ,  $\min\{x_m(T), y_m(T)\} \geq \alpha_m/(9\sqrt{m})$ , and  $h_m(T) \geq \alpha_m/(9m)$ .*

In general, one cannot omit the condition  $\lim \alpha_m = 0$ . Indeed, suppose  $X = \ell_p$  (or  $X = c_0$ ), and  $Y = \ell_q$ , with  $p > q \geq 1$  ( $\infty > q \geq 1$  if  $X = c_0$ ). By Pitt's theorem [14, Proposition 2.c.3], any  $T \in B(X, Y)$  is compact. Furthermore,  $Y$  has the Approximation Property, hence, by [14, Theorem 1.e.4], any compact operator into  $Y$  is approximable. Thus,  $\lim_m a_m(T) = 0$  for any  $T \in B(X, Y)$ .

The lower estimates for  $x_m(T)$ ,  $y_m(T)$ , and  $h_m(T)$  are best possible, too.

**Proposition 1.2.**  $\lim \sqrt{k}x_k(T) = \lim \sqrt{k}y_k(T) = \lim kh_k(T) = 0$  for any  $T \in B(c_0, \ell_1)$ .

Note that, unlike the results of [1, 8], Theorem 1.1 covers all pairs  $(X, Y)$  of infinite dimensional Banach spaces. We do not know whether this theorem can be strengthened to obtain  $T \in B(X, Y)$  with (say)  $\alpha_i \leq a_i(T) \leq C\alpha_i$ , for some fixed constant  $C$ . However, for some pairs  $(X, Y)$ , one cannot find an operator  $T : X \rightarrow Y$  with *precisely* the prescribed Gelfand or approximation numbers. Recall that a Banach space  $X$  is called *strictly convex* if for every  $x, y \in X$ ,  $\|x + y\| = \|x\| + \|y\|$  can hold only if  $x$  and  $y$  are scalar multiples of each other (see e.g. [9]). Therefore, any  $x^* \in X^*$  can attain its norm at no more than one point of the unit ball of  $X$ . It is known that for every separable Banach space there exists an equivalent strictly convex norm (and more – see Section 1 of [5]).

**Proposition 1.3.** *Suppose  $X$  is a strictly convex reflexive Banach space, and  $T : X \rightarrow c_0$  is compact. Then  $a_2(T) = c_2(T) < a_1(T) = c_1(T) = \|T\|$ .*

The condition that  $T$  is compact (equivalently,  $\lim_k c_k(T) = 0$ ) is essential: if  $T$  is the formal embedding of  $\ell_p$  to  $c_0$  ( $1 \leq p < \infty$ ), then  $c_k(T) = 1$  for each  $k$ . However, the compactness of  $T$  is equivalent to  $\lim a_i(T) = 0$ .

Note also that  $\ell_1$  is not strictly convex, hence the above proposition doesn't apply to the operators from  $\ell_1$  to  $c_0$ . In fact, [1] shows that, for any decreasing sequence  $(\alpha_m)$  with  $\lim \alpha_m = 0$ , there exists  $T \in B(\ell_1, c_0)$  such that  $a_m(T) = \alpha_m$  for every  $m$ .

Now suppose  $\mathcal{A}$  is a quasi-Banach operator ideal, equipped with the norm  $\|\cdot\|_{\mathcal{A}}$  (see e.g. [4, 22, 26] for the definition and basic properties of operator ideals). Define the  $\mathcal{A}$ -approximation numbers by setting

$$a_n^{(\mathcal{A})}(T) = \inf_{u \in B(X, Y), \text{rank } u < n} \|T - u\|_{\mathcal{A}}.$$

The  $\mathcal{A}$ -Gelfand numbers are defined by

$$c_n^{(\mathcal{A})}(T) = \inf_{E \hookrightarrow X, \text{codim } E < n} \|T|_E\|_{\mathcal{A}}.$$

We are especially interested in the ideals of  $p$ -factorable and  $(t, r)$ -summing operators. Recall that  $T \in B(X, Y)$  is called  $p$ -factorable ( $1 \leq p \leq \infty$ ) if it can be represented as  $T = T_2 T_1$ , with  $T_1 \in B(X, L_p(\mu))$  and  $T_2 \in B(L_p(\mu), Y)$ . The associated norm is given by  $\gamma_p(T) = \inf \|T_2\| \|T_1\|$ , with the infimum running over all representations of the above form. The ideal of all  $p$ -summing operators is denoted by  $\Gamma_p$ .

An operator  $T \in B(X, Y)$  is  $(t, r)$ -summing ( $1 \leq r \leq t \leq \infty$ ) if there exists a constant  $C$  such that, for every  $x_1, \dots, x_n \in X$ ,

$$\left( \sum_{i=1}^n \|Tx_i\|^t \right)^{1/t} \leq C \left( \sup_{x^* \in X^*, \|x^*\| \leq 1} \sum_{i=1}^n |\langle x^*, x_i \rangle|^r \right)^{1/r}.$$

The infimum of all  $C > 0$  with the above property is denoted by  $\pi_{t,r}(T)$ , and the corresponding ideal – by  $\Pi_{tr}$ . When  $t = r$ , we use the notation  $\pi_r$  and  $\Pi_r$ , and the term  $r$ -summing.

For  $p$ -factorable operators, we have:

**Theorem 1.4.** *For  $1 < p < \infty$ , there exists a constant  $K_p$  such that, whenever  $X$  and  $Y$  are infinite dimensional Banach spaces, and  $\alpha_k \searrow 0$ , there exists an approximable  $T : X \rightarrow Y$  such that  $\|T\| \leq 2\alpha_1$  and, for every  $m$ ,*

$$K_p \alpha_{\lceil m/6 \rceil} \geq a_m^{(\Gamma_p)}(T) \geq c_m^{(\Gamma_p)}(T) \geq \alpha_m/9.$$

As we shall see below, the operator  $T$  constructed in Theorem 1.1 has the properties described by this theorem.

Next we handle the ideal of  $p$ -summing operators  $\Pi_p$ .

**Theorem 1.5.** *If  $X$  and  $Y$  are infinite dimensional Banach spaces, and  $\alpha_k \searrow 0$ , there exists a 2-summing map  $T \in B(X, Y)$ , such that*

$$c_p \alpha_{18m} \leq c_m^{(\Pi_p)}(T) \leq a_m^{(\Pi_p)}(T) \leq 3\alpha_{\lceil 4m/5 \rceil}$$

for every  $m$ , and every  $p \in [2, \infty)$  ( $c_p$  is a constant depending on  $p$ ).

For certain pairs  $(X, Y)$ , one can construct  $T \in B(X, Y)$  with the prescribed rate of decay of  $(c_m^{(\mathcal{A})}(T))$  and  $(a_m^{(\mathcal{A})}(T))$  for other classes of ideals  $\mathcal{A}$ . Recall that a Banach operator ideal  $\mathcal{A}$  is called 1-*injective* if, for any  $u \in B(X, Y)$ , and any isometric injection  $J : Y \rightarrow Y_0$ , we have  $\|u\|_{\mathcal{A}} = \|Ju\|_{\mathcal{A}}$ . For instance, the ideal  $\Pi_{tr}$  of  $(t, r)$ -summing operators is 1-injective.

**Theorem 1.6.** *Suppose  $\alpha_k \searrow 0$ , and the Banach spaces  $X$  and  $Y$  have no non-trivial cotype and no non-trivial type, respectively. Then there exists  $T \in B(X, Y)$  such that*

$$\frac{1}{50}\alpha_{18m} \leq c_m(T) \leq c_m^{(\mathcal{A})}(T) \leq a_m^{(\mathcal{A})}(T) \leq 4\alpha_{\lceil 4m/5 \rceil}$$

for every  $m$ , and every 1-injective Banach operator ideal  $\mathcal{A}$ .

We shall say that a Banach space  $X$  has *Property  $(\mathcal{P})_C$*  ( $C \geq 1$ ) if, for any  $n \in \mathbb{N}$ , and any finite codimensional  $X' \hookrightarrow X$ , there exists an  $n$ -dimensional  $E \hookrightarrow X'$  such that  $d(E, \ell_2^n) \leq C$ , and  $E$  is  $C$ -complemented in  $X$ . By [19], any space with non-trivial type has Property  $(\mathcal{P})_C$ , for some  $C$ . Consequently, any Banach space containing a complemented subspace of non-trivial type has Property  $(\mathcal{P})_C$  for some  $C$ .

**Theorem 1.7.** *Suppose an infinite dimensional Banach space  $X$  has Property  $(\mathcal{P})_C$ , for some  $C > 1$ . Then, for any infinite dimensional Banach space  $Y$ , and any sequence  $\alpha_k \searrow 0$ , there exists  $T \in B(X, Y)$  such that*

$$\frac{1}{20C^2}\alpha_{18m} \leq c_m^{(\Pi_{tr})}(T) \leq a_m^{(\Pi_{tr})}(T) \leq 4\alpha_{\lceil 4m/5 \rceil}$$

for every  $m \in \mathbb{N}$ , and for any  $t$  and  $r$  satisfying  $1 \leq r \leq \min\{2, t\}$ , and  $1/r - 1/t < 1/2$ .

We do not know how well one can control the rate of decay of  $(a_i^{(\mathcal{A})}(T))$  for general ideals  $\mathcal{A}$ . It was shown in [2] that, for any quasi-Banach (respectively, Banach) ideal  $\mathcal{A}$ , and every sequence  $\alpha_i \searrow 0$ , there exists  $T \in B(X, Y)$  such that  $\lim a_i^{(\mathcal{A})}(T) = 0$ , and  $a_i^{(\mathcal{A})}(T) \geq \alpha_i$  for infinitely many (respectively, all) values of  $i$ .

We prove the results stated above in Section 2. Throughout, we assume  $\alpha_1 > 0$  (the case of  $\alpha_1 = 0$  is trivial). We use the common Banach and operator space notation (see e.g. [4, 14]).  $B(X)$  denotes the closed unit ball of  $X$ .  $d(E, F)$  stands for the *Banach-Mazur distance* between Banach spaces  $E$  and  $F$ . That is,  $d(E, F) = \inf \|u\| \|u^{-1}\|$ , with the infimum running over all invertible maps  $u : E \rightarrow F$ .

## 2. PROOFS

The proofs of some of the results requires using copies of  $\ell_2^n$  as building blocks (in a way reminiscent of [21]). We thus need:

**Lemma 2.1.** *Suppose  $c > 1$ ,  $(n_k)$  is a sequence of positive integers, and  $\Gamma$  is an infinite set.*

- (1) *Suppose  $J$  is an isometric injection of an infinite dimensional Banach space  $Y$  to  $\ell_\infty(\Gamma)$ . Then there exist subspaces  $(F_k)$  of  $Y$ , and finite rank maps  $v_k \in B(\ell_\infty(\Gamma))$ , such that: (i)  $d(F_k, \ell_2^{n_k}) < \sqrt{c}$ , (ii)  $\|v_k\| < c + 1$ , (iii)  $v_k J j_k = J j_k$ , (iv) for  $s \neq k$ ,  $v_s J j_k = 0$  ( $j_k$  denotes the canonical inclusion of  $F_k$  into  $Y$ ).*

- (2) Suppose  $X$  is an infinite dimensional Banach space, and  $Q : \ell_1(\Gamma) \rightarrow X$  is a quotient map. Then there exist quotients  $E_k$  of  $X$  ( $q_k : X \rightarrow E_k$  is the quotient map) and weak\* continuous maps  $u_k \in B(\ell_\infty(\Gamma))$ , such that (i)  $d(E_k, \ell_2^{n_k}) < \sqrt{c}$ , (ii)  $\|u_k\| < c + 1$ , (iii)  $u_k|_{Q^*q_k^*(E_k^*)} = I_{Q^*q_k^*(E_k^*)}$ , and (iv) for  $s \neq k$ ,  $u_s Q^* q_k^* = 0$ .

*Proof.* (1) Select  $\lambda \in (1, \sqrt{c})$  in such a way that  $\lambda(1 + \lambda) < 1 + c$ . We construct the spaces  $F_k$  and operators  $v_k$  recursively. By Dvoretzky's Theorem, there exists  $F_1 \hookrightarrow Y$  such that  $d(F_1, \ell_2^{n_1}) < \sqrt{c}$ . Furthermore, we can find a finite rank projection  $v_1 \in B(\ell_\infty(\Gamma))$  such that  $v_1|_{F_1} = I_{F_1}$ , and  $\|v_1\| < c$ .

Now suppose  $F_1, \dots, F_{k-1}, v_1, \dots, v_{k-1}$  with the desired properties have been constructed. Find a finite rank projection  $P_1 \in B(\ell_\infty(\Gamma))$  such that  $\|P_1\| < \lambda$ , and  $P_1 v_s = v_s$  for  $1 \leq s < k$ . Find  $F_k \hookrightarrow Y \cap \ker P_1 \cap (\cap_{s=1}^{k-1} \ker v_s)$ , such that  $d(F_k, \ell_2^{n_k}) < \sqrt{c}$ . Finally, find a finite rank projection  $P_2 \in B(\ell_\infty(\Gamma))$  such that  $\|P_2\| < \lambda$ , and  $P_2|_{F_k} = I_{F_k}$ . Let  $v_k = P_2(I - P_1)$ . Then  $v_k|_{F_k} = I_{F_k}$ . By our choice of  $\lambda$ ,  $\|v_k\| < 1 + c$ . For  $s < k$ , we have  $v_k v_s = 0$ , and  $v_s|_{F_k} = 0$ . Thus,  $v_s J j_k = 0$  if  $s \neq k$ .

(2) By (1), there exist subspaces  $G_1, G_2, \dots$  of  $X^*$ , and finite rank operators  $w_k \in B(\ell_\infty(\Gamma))$ , such that  $d(G_k, \ell_2^{n_k}) < \sqrt{c}$ ,  $\|w_k\| < c + 1$ ,  $w_k|_{G_k} = I_{G_k}$ ,  $w_k w_s = 0$  for  $s < k$ , and  $w_s|_{G_k} = 0$  for  $s > k$ . By [18, Theorem 2.5], there exists a sequence of finite rank maps  $u_k \in B(\ell_1(\Gamma))$  such that  $\|u_k\| < c + 1$ ,  $\text{ran } u_k^* = \text{ran } w_k$ , and  $(u_k^* - w_k)|_{G_k \cup (\cup_{s < k} \text{ran } w_s)} = 0$ . Furthermore, the isometric embedding  $i_k : G_k \rightarrow X^*$  is the dual of the quotient map  $q_k : X \rightarrow E_k = G_k^*$ , where  $q_k = i_k^*|_X$ .  $\blacksquare$

*Proof of Theorem 1.1.* Select a set  $\Gamma$  for which there exist an isometric embedding  $J : Y \rightarrow \ell_\infty(\Gamma)$ , and a quotient map  $Q : \ell_1(\Gamma) \rightarrow X$ . Set  $n_0 = 0$ , and find a sequence  $(n_k) \subset \mathbb{N}$  such that, for each  $k$ , (i)  $n_k > 5(n_{k-1} + 1)$ , and (ii)  $\alpha_{n_k} \leq \alpha_{n_{k-1}+1}/5$ . Select  $c > 1$  such that  $c^2(1 + c)^2 < 9/2$ . By Lemma 2.1, there exist embeddings  $j_k : F_k \rightarrow Y$ , quotient maps  $q_k : X \rightarrow E_k$ , and finite rank operators  $u_k, v_k \in B(\ell_\infty(\Gamma))$ , such that, for each  $k$ :

- $\max\{d(E_k, \ell_2^{n_k}), d(F_k, \ell_2^{n_k})\} < \sqrt{c}$ .
- $\max\{\|u_k\|, \|v_k\|\} < c + 1$ .
- $u_k$  is weak\* continuous (hence  $u_k^*$  maps  $\ell_1(\Gamma) = \ell_\infty(\Gamma)_*$  into itself).
- $u_k Q^* q_k^* = Q^* q_k^*$  (equivalently,  $q_k Q u_k^* = q_k Q$ ), and  $v_k J j_k = J j_k$ .
- For  $s \neq k$ ,  $u_s Q^* q_k^* = 0$ , and  $v_s J j_k = 0$ .

For each  $k$ , find contractions  $U_k : E_k \rightarrow \ell_2^{n_k}$  and  $V_k : \ell_2^{n_k} \rightarrow F_k$ , such that their inverses have norms smaller than  $\sqrt{c}$ . For  $1 \leq j \leq n_k$ , set  $\beta_{jk} = \min\{\alpha_{n_{k-1}+1}, \alpha_j\}$ . Denote the canonical basis in  $\ell_2^{n_k}$  by  $(\delta_{jk})_{j=1}^{n_k}$ , and define the diagonal operator  $D_k \in B(\ell_2^{n_k})$  by setting  $D_k \delta_{jk} = \beta_{jk} \delta_{jk}$  ( $1 \leq j \leq n_k$ ). Let  $S_k = V_k D_k U_k$ . Then  $\|S_k\| \leq \alpha_{n_{k-1}+1}$ , hence  $T = \sum_{s=1}^\infty j_s S_s q_s$  is approximable, and  $\|T\| < 2\alpha_1$ .

To estimate  $t_m(T) = a_m(JTQ)$  from below, find  $k$  satisfying  $n_{k-1} < m \leq n_k$ . Recall that  $v_k J j_k = J j_k$ ,  $q_k Q u_k^* = q_k Q$ , and, for  $s \neq k$ ,  $v_k J j_s$  and  $q_s Q u_k^*$  vanish. Therefore, for any  $s$ -scale,

$$(2.1) \quad (1 + c)^2 s_m(JTQ) \geq s_m(v_k JTQ u_k^*) = s_m\left(\sum_s v_k J j_s S_s q_s Q u_k^*\right) = s_m(J j_k S_k q_k Q).$$

Consequently,

$$(2.2) \quad t_m(T) = a_m(JTQ) \geq (1+c)^{-2} a_m(Jj_k S_k q_k Q).$$

To proceed further, note that  $q_k Q(B(\ell_1(\Gamma))) = B(E_k)$ , and  $Jj_k|_{\text{ran } S_k} = I_{\text{ran } S_k}$ . Let  $G = V_k(\text{span}[\delta_{jk} : j \leq m])$ . Then  $Jj_k S_k q_k Q(B(\ell_1(\Gamma)))$  contains  $c^{-1} \alpha_m B(G)$ , and therefore (see e.g. Lemma 1.19 of [7]),

$$a_m(Jj_k S_k q_k Q) \geq d_m(Jj_k S_k q_k Q) \geq c^{-1} \alpha_m.$$

Together with (2.2), this yields the desired estimate for  $t_m(T)$ .

Next we estimate  $a_m(T)$  from above. Let  $n'_k = n_1 + \dots + n_{k-1}$  (by our assumption on the sequence  $(n_j)$ ,  $n'_k < 4n_{k-1}/3$ ). Assume first that  $m > 3n'_k/2$ . Write  $T = T^{(1)} + T^{(2)} + T^{(3)}$ , where

$$T^{(1)} = \sum_{s=1}^{k-1} Jj_s S_s q_s Q, \quad T^{(2)} = Jj_k S_k q_k Q, \quad \text{and} \quad T^{(3)} = \sum_{s=k+1}^{\infty} Jj_s S_s q_s Q.$$

Then

$$a_m(T) \leq a_m(T^{(1)} + T^{(2)}) + \|T^{(3)}\| \leq a_{m-\text{rank } T^{(1)}}(T^{(2)}) + \|T^{(3)}\|.$$

Then  $\|T^{(3)}\| \leq \sum_{s=k}^{\infty} \|S_{s+1}\| \leq \sum_{s=k}^{\infty} \alpha_{n_s+1}$ . But, for  $j \geq 0$ ,  $\alpha_{n_{k+j}+1} \leq 5^{-j} \alpha_{n_k} \leq 5^{-j} \alpha_m$ , hence  $\|T^{(3)}\| \leq 5\alpha_m/4$ . Furthermore,  $\text{rank } T^{(1)} \leq n'_k$ . Therefore, by [24],

$$a_{m-n'_k}(T^{(2)}) \leq a_{m-n'_k}(D_k) \leq \beta_{m-n'_k, k} = \alpha_{\max\{m-n'_k, n_{k-1}+1\}} \leq \alpha_{\lceil m/3 \rceil},$$

hence  $a_m(T) \leq 3\alpha_{\lceil m/3 \rceil}$ .

Now suppose  $n_{k-1} < m \leq 3n'_k/2$ . As  $n_{k-1} > n'_{k-1}$ , the reasoning above shows  $a_m(T) \leq a_{n_{k-1}}(T) \leq 3\alpha_{\lceil n_{k-1}/3 \rceil}$ . Furthermore,  $m \leq 3n'_k/2 < 2n_{k-1}$ , hence  $a_m(T) \leq 3\alpha_{\lceil m/6 \rceil}$ .

Before establishing lower estimates for other  $s$ -numbers mentioned in the theorem, recall a few known facts. By [11], any linear operator  $u : Z_0 \rightarrow G$  ( $G$  is a finite dimensional space with  $\dim G > 1$ , and  $Z_0$  is a subspace of a Banach space  $Z$ ) has an extension  $\tilde{u} : Z \rightarrow G$ , satisfying  $\|\tilde{u}\| < \sqrt{\dim G} \|u\|$ . Moreover, by [18],  $\tilde{u}$  can be taken to weak\* continuous if  $Z_0$  is finite dimensional.

To estimate  $x_m(T)$ , pick  $k$  with  $n_{k-1} < m \leq n_k$ . We consider the case of  $m > 1$ , as  $x_1(T)$  can be estimated similarly. By (2.1),  $x_m(T) \geq (1+c)^{-2} x_m(Jj_k S_k q_k Q)$ . Let  $H = \text{span}[\delta_{jk} : 1 \leq k \leq m]$ , and  $E = U_k^{-1}(H)$ . We find a contraction  $a : \ell_2^m \rightarrow \ell_1(\Gamma)$ , for which

$$(2.3) \quad q_k Q a(B(\ell_2^m)) \subset (cm)^{-1/2} B(E).$$

Once we have such an  $a$ , recall that  $\|D_k \xi\| \geq \alpha_m \|\xi\|$  for any  $\xi \in H$ . Therefore,

$$Jj_k S_k q_k Q a(B(\ell_2^m)) \supset c^{-3/2} m^{-1/2} \alpha_m B(V_k(H)),$$

hence

$$x_m(Jj_k S_k q_k Q) \geq d_m(Jj_k S_k q_k Q a) \geq c^{-3/2} m^{-1/2} \alpha_m.$$

To construct  $a$  as above, denote the inclusion of  $E$  into  $E_k$  by  $i$ . Recall that  $Q^* q_k^*$  is an isometric embedding of  $E_k^*$  into  $\ell_\infty(\Gamma)$ . As noted above,  $i^* : E_k^* \rightarrow E^*$  has a weak\*

continuous extension  $a_0^* : \ell_\infty(\Gamma) \rightarrow E^*$ , such that  $\|a_0\| < \sqrt{m}$ . Then  $a = (cm)^{-1/2}a_0U_k^{-1}|_H$  satisfies (2.3).

To handle  $y_m(T)$  and  $h_m(T)$ , we need a contraction  $b : \ell_\infty(\Gamma) \rightarrow \ell_2^m$  such that  $\sqrt{cm}bJj_k = V_k^{-1}|_{V_k(H)}$  (here, we identify  $H$  with  $\ell_2^m$ ). To show that such a  $b$  exists, note that the operator  $V_k^{-1} : V_k(H) \rightarrow H$  has an extension  $b_0 : \ell_\infty(\Gamma) \rightarrow H$ , with  $\|b_0\| < \sqrt{cm}$ . Then  $b = (cm)^{-1/2}b_0$  has the desired properties.

By (2.1),  $y_m(T) \geq (1+c)^{-2}a_m(bJj_kS_kq_kQ)$ . Recall that  $\|D_k\xi\| \geq \alpha_m\|\xi\|$  for any  $\xi \in H$ . Thus,  $bJj_kS_kq_kQ(B(\ell_1(\Gamma))) \supset c^{-3/2}m^{-1/2}\alpha_mB(V_k(H))$ . By (2.1),

$$y_m(T) \geq (1+c)^{-2}a_m(bJj_kS_kq_kQ) \geq c^{-3/2}(1+c)^{-2}m^{-1/2}\alpha_m.$$

Furthermore,  $bJj_kS_kq_kQa(B(\ell_2^m))$  contains  $c^{-2}m^{-1}\alpha_mB(\ell_2^m)$ , hence

$$h_m(T) \geq (1+c)^{-2}a_m(bJj_kS_kq_kQa) \geq m^{-1}c^{-2}(1+c)^{-2}\alpha_m.$$

As  $c^2(1+c)^2 < 9/2$ , we are done. ■

*Proof of Proposition 1.2.* Denote by  $P_N$  the projection onto the span of the first  $N$  elements of the canonical basis in  $c_0$ . Then  $P_N^*$  is the projection onto the span of the first  $N$  elements of the canonical basis in  $\ell_1$ . First show that, for  $T \in B(c_0, \ell_1)$ ,

$$(2.4) \quad \lim_N \|T - P_N^*TP_N\| = 0$$

As noted in the paragraph preceding the statement of this theorem, for every  $\varepsilon > 0$  there exists a finite rank operator  $S$  satisfying  $\|T - S\| < \varepsilon/3$ . Write  $S = \sum_{i=1}^n y_i \otimes z_i$ , with  $y_i, z_i \in \ell_1$  (that is, for  $x \in c_0$ ,  $Sx = \sum_{i=1}^n \langle y_i, x \rangle z_i$ ). Then  $P_N^*SP_Nx = \sum_{i=1}^n \langle P_N^*y_i, x \rangle P_N^*z_i$ . Note that  $\lim_N P_N^*y = y$  for any  $y \in \ell_1$ , hence there exists  $M \in \mathbb{N}$  with  $\|P_N^*SP_N - S\| < \varepsilon/3$  for any  $N \geq M$ . For such values of  $N$ ,

$$\|T - P_N^*TP_N\| \leq \|T - S\| + \|S - P_N^*SP_N\| + \|P_N^*(S - T)P_N\| < \varepsilon.$$

As  $\varepsilon$  is arbitrary, (2.4) follows.

Recall that, for any  $s$ -scale,  $s_m(u+v) \geq s_{m-\text{rank } v}(u)$  if  $v$  is a finite rank operator, and  $m \geq \text{rank } v$ . In the above notation,  $\text{rank}(P_N^*TP_N) \leq N$ , hence

$$(2.5) \quad s_m(T) \leq s_{m-N}(T - P_N^*TP_N).$$

We study  $(x_k(\cdot))$  first. By Grothendieck Theorem,  $\pi_2(u) \leq K_G\|u\|$  for any  $u \in B(c_0, \ell_1)$ . Furthermore, for any operator  $u$  and  $k \in \mathbb{N}$ ,  $x_k(u) \leq \pi_2(u)/\sqrt{k}$  [12, Lemma 9]. Thus, for any  $u \in B(c_0, \ell_1)$  and  $k \in \mathbb{N}$ ,

$$(2.6) \quad x_k(u) \leq K_G\|u\|/\sqrt{k}.$$

Now fix  $T \in B(c_0, \ell_1)$ . For any  $\varepsilon > 0$ , there exists  $M \in \mathbb{N}$  such that  $\|T - P_N^*TP_N\| < \varepsilon$  for any  $N \geq M$ . Applying (2.6) to  $u = T - P_N^*TP_N$ , and invoking (2.5), we conclude that  $x_k(T) \leq x_{k-N}(T - P_N^*TP_N) \leq K_G\varepsilon/\sqrt{k-N}$  for every  $k > N$ . Thus,

$$\limsup_k \sqrt{k}x_k(T) \leq \lim_k \sqrt{k/(k-N)}K_G\varepsilon = K_G\varepsilon.$$

As  $\varepsilon > 0$  is arbitrary, we conclude that  $\lim \sqrt{k}x_k(T) = 0$ .



To establish  $\lim \sqrt{k}y_k(T) = 0$ , recall that  $s_m(u + v) \leq s_m(u) + \|v\|$  for any operators  $u$  and  $v$ , and any  $s$ -scale  $(s_m)$ . In particular, for any  $S \in B(c_0, \ell_1)$ ,  $y_k(S) \leq y_k(P_M^*SP_M) + \|S - P_M^*SP_M\|$  for any  $M \in \mathbb{N}$ . By duality,  $y_k(u) = x_k(u^*)$  if  $u$  is an operator between finite dimensional spaces. Viewing  $P_M^*SP_M$  as an element of  $B(\ell_\infty^M, \ell_1^M)$ , and applying (2.6), we obtain  $y_k(P_M^*SP_M) = x_k((P_M^*SP_M)^*) \leq K_G\|S\|/\sqrt{k}$ . As  $\lim_M \|S - P_M^*SP_M\| = 0$ , we conclude that  $y_k(S) \leq K_G\|S\|/\sqrt{k}$ .

Using the inequality from the previous paragraph with  $S = T - P_N^*TP_N$  ( $T \in B(c_0, \ell_1)$ ,  $k > N \in \mathbb{N}$ ), and invoking (2.5), we conclude that  $y_k(T) \leq K_G\|T - P_N^*TP_N\|/\sqrt{k - N}$ . Applying (2.4) (as in the case of  $x_k(T)$ ) yields  $\lim_k \sqrt{k}y_k(T) = 0$ .

Finally we tackle  $(h_k(\cdot))$ . Show first that, for any  $S \in B(c_0, \ell_1)$ , and any two contractions  $u : \ell_2 \rightarrow c_0$  and  $v : \ell_1 \rightarrow \ell_2$ , we have  $a_k(vSu) \leq K_G^2\|S\|/k$ . To achieve this, denote the nuclear norm of an operator by  $\nu(\cdot)$ . By [20, Sections 1 and 5],

$$\nu(vSu) \leq \pi_2(v)\pi_2(S)\|u\| \leq K_G^2\|v\|\|S\|\|u\| = K_G^2\|S\|.$$

Denoting the singular numbers of  $vSu$  by  $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$ , we see that  $\nu(vSu) = \lambda_1 + \lambda_2 + \dots \leq K_G^2\|S\|$ , and  $a_k(vSu) = \lambda_k$ . Clearly,  $\lambda_k \leq \nu(vSu)/k \leq K_G^2\|S\|/k$ .

Now consider  $T \in B(c_0, \ell_1)$ . For any  $\varepsilon > 0$ , there exists  $M \in \mathbb{N}$  such that  $\|T - P_N^*TP_N\| < \varepsilon$  for  $N \geq M$ . Combining the previous paragraph with (2.5), we see that, for  $k > N$ ,  $a_k(T) \leq a_{k-N}(T - P_N^*TP_N) \leq K_G^2\varepsilon/(k - N)$ . Thus,  $\limsup_k ka_k(T) \leq K_G^2\varepsilon$ . As  $\varepsilon > 0$  is arbitrary, the proof is complete.  $\blacksquare$

*Proof of Proposition 1.3.* Note first that, if  $Y$  is an  $L_1$  predual, and  $T : X \rightarrow Y$  is a compact operator, then  $c_k(T) = a_k(T)$  for any  $k$ . Indeed, fix  $\varepsilon > 0$ , and find  $E \hookrightarrow X$  such that  $\dim X/E < k$ , and  $\|T|_E\| < c_k(T) + \varepsilon/2$ . By [13], there exists  $S : X \rightarrow Y$  such that  $S|_E = T|_E$ , and  $\|S\| < c_k(T) + \varepsilon$ . Let  $u = T - S$ . Then  $\text{rank } u < k$ , and  $a_k(T) \leq \|T - u\| = \|S\| < c_k(T) + \varepsilon$ . As  $\varepsilon$  is arbitrary, we are done.

Thus, it suffices to show the non-existence of a  $T \in K(X, c_0)$  with  $\|T\| = c_2(T) = 1$ . Suppose, for the sake of contradiction, that such a  $T$  exists. Then there exists a unique sequence  $(x_i^*)_{i \in \mathbb{N}} \in c_0(X^*)$  such that  $\max_i \|x_i^*\| = \|T\| = 1$ , and  $Tx = (\langle x, x_i^* \rangle)_{i \in \mathbb{N}}$  for every  $x \in X$ . Let  $N = \max\{i : \|x_i^*\| = 1\}$ . If  $c_2(T) = 1$ , then, for every 1-codimensional  $E \hookrightarrow X$ ,

$$\max_{1 \leq i \leq N} \sup_{x \in E, \|x\| \leq 1} |\langle x_i^*, x \rangle| = 1.$$

By the reflexivity of  $X$ , the sup in the centered expression is attained. Therefore, for every such  $E$  there exists  $i \in \{1, \dots, N\}$  such that  $E$  contains  $E_i$ , where  $E_i$  is the (one-dimensional) linear span of the unique  $x_i \in X$  satisfying  $\|x_i\| = 1 = \langle x_i^*, x_i \rangle$ . In other words, any  $x^* \in X^*$  satisfies  $\langle x^*, x_i \rangle = 0$ , for some  $i$ . This, however, is impossible.  $\blacksquare$

*Proof of Theorem 1.4.* We re-use the operator  $T$  constructed in the proof of Theorem 1.1, and the notation introduced there. The desired lower estimate follows from  $c_m^{(\Gamma_p)}(T) \geq c_m(T)$ . To estimate  $a_m^{(\Gamma_p)}(T)$  from above, assume first  $m > 3n'_k/2$ , where  $n'_k = n_1 + \dots +$



$n_{k-1}$ . Write  $T = T^{(1)} + T^{(2)} + T^{(3)}$ , where

$$T^{(1)} = \sum_{s=1}^{k-1} j_s S_s q_s, \quad T^{(2)} = j_k S_k q_k, \quad \text{and} \quad T^{(3)} = \sum_{s=k+1}^{\infty} j_s S_s q_s.$$

Then

$$a_m^{(\Gamma_p)}(T) \leq a_m^{(\Gamma_p)}(T^{(1)} + T^{(2)}) + \gamma_p(T^{(3)}) \leq a_{m-\text{rank } T^{(1)}}^{(\Gamma_p)}(T^{(2)}) + \sum_{s=k}^{\infty} \gamma_p(D_{s+1}).$$

Note that  $L_p$  contains a  $C_p$ -complemented copy of  $L_2$  (with  $C_p \sim \max\{\sqrt{p}, 1/\sqrt{p-1}\}$ ), hence  $\gamma_p(D_{s+1}) \leq C_p \|D_{s+1}\| \leq C_p \alpha_{n_{s+1}}$ . But, for  $j \geq 0$ ,  $\alpha_{n_{k+j}+1} \leq 5^{-j} \alpha_{n_k+1} \leq 5^{-j} \alpha_m$ , hence  $\sum_{s=k}^{\infty} \gamma_p(D_{s+1}) \leq 5C_p \alpha_m/4$ . Furthermore,  $\text{rank } T^{(1)} \leq n'_k$ . Therefore,

$$a_{m-n'_k}^{(\Gamma_p)}(T^{(2)}) \leq a_{m-n'_k}^{(\Gamma_p)}(D_k) \leq C_p \beta_{m-n'_k, k} = C_p \alpha_{\max\{m-n'_k, n_{k-1}+1\}} \leq C_p \alpha_{\lceil m/2 \rceil},$$

hence  $a_m^{(\Gamma_p)}(T) \leq 3C_p \alpha_{\lceil m/3 \rceil}$ .

Finally, we handle the case of  $n_{k-1} < m \leq 3n'_k/2$  as in Theorem 1.1. ■

The proof of Theorem 1.5 requires a technical result, which may be known to experts. We say that a sequence  $(\alpha_k)_{k \in \mathbb{N}}$  is *convex* if

$$\alpha_k \leq \frac{n-k}{n-m} \alpha_m + \frac{k-m}{n-m} \alpha_n$$

whenever  $m < k < n$ . It is easy to see that, for any non-increasing convex sequence  $(\alpha_k)$  of non-negative numbers,

$$(2.7) \quad \frac{\alpha_i - \alpha_j}{j-i} \geq \frac{\alpha_m - \alpha_n}{n-m}$$

if  $j > i$ ,  $n > m$ ,  $i \leq m$ , and  $j \leq n$ .

**Lemma 2.2.** *Suppose  $(\alpha_k)_{k \in \mathbb{N}}$  is a non-increasing sequence, converging to 0. Then there exists a convex sequence  $(\beta_k)_{k \in \mathbb{N}}$  satisfying  $\alpha_k \geq \beta_k \geq \min\{\alpha_k/2, \alpha_{2k-1}\}$  for any  $k$ .*

*Proof.* Set  $\beta_1 = \alpha_1$ . For  $k > 1$ , define

$$\beta_k = \inf_{m \leq k \leq n, m < n} \left\{ \frac{n-k}{n-m} \alpha_m + \frac{k-m}{n-m} \alpha_n \right\}.$$

The standard “convex envelope” arguments (see e.g. [15, p. 66]) show that  $(\beta_k)$  is indeed a convex sequence. Thus, it suffices to show that

$$(2.8) \quad \frac{n-k}{n-m} \alpha_m + \frac{k-m}{n-m} \alpha_n \geq \min\{\alpha_k/2, \alpha_{2k-1}\}$$

if  $m \leq k \leq n$  and  $m < n$ . If  $n < 2k$ , then

$$\frac{n-k}{n-m} \alpha_m + \frac{k-m}{n-m} \alpha_n \geq \frac{n-k}{n-m} \alpha_n + \frac{k-m}{n-m} \alpha_n \geq \alpha_{2k-1}.$$

On the other hand, if  $n \geq 2k$ , then  $(n-k)/(n-m) > 1/2$ , and

$$\frac{n-k}{n-m} \alpha_m + \frac{k-m}{n-m} \alpha_n \geq \frac{n-k}{n-m} \alpha_m > \frac{\alpha_k}{2}.$$

In either case, (2.8) holds. ■

We also need to be able to estimate  $p$ -summing norms of diagonal operators.

**Lemma 2.3.** *For  $p \in [2, \infty)$ , there exists  $\kappa_p \in (0, 1]$  ( $\kappa_2 = 1$ ) such that:*

- (1) *If  $u$  is an operator on a Hilbert space, then  $\kappa_p \|u\|_{HS} \leq \pi_p(u) \leq \|u\|_{HS}$*
- (2) *If  $D = \text{diag}(d_i)_{i=1}^N$  is a diagonal operator from  $\ell_\infty^N$  to  $\ell_2^N$ , then  $\kappa_p (\sum_i |d_i|^2)^{1/2} \leq \pi_p(D) \leq (\sum_i |d_i|^2)^{1/2}$ .*

*Proof.* Part (1) can be found in e.g. [4]. Part (2) is also known. We provide the proof for the sake of completeness. By scaling, we can assume that  $\sum_i |d_i|^2 = 1$ .

Consider the case of  $p = 2$  first. Pietsch Factorization Theorem yields  $\pi_2(D) \leq 1$ . On the other hand, let  $(e_i)_{i=1}^N$  be the canonical basis for  $\ell_\infty^N$ . Then

$$\left( \sum_{i=1}^N \|De_i\|^2 \right)^{1/2} = 1 = \sup_{f \in \ell_1^N, \|f\|=1} \left( \sum_{i=1}^N |\langle f, e_i \rangle|^2 \right)^{1/2},$$

hence  $\pi_2(D) \geq 1$ . Thus,  $\pi_2(D) = (\sum_i |d_i|^2)^{1/2}$ .

Now suppose  $p > 2$ . Trivially,  $\pi_p(D) \leq \pi_2(D) = (\sum_i |d_i|^2)^{1/2}$ . To prove the opposite inequality, denote by  $id$  the formal identity map from  $\ell_2^N$  to  $\ell_\infty^N$ . By Part (1),

$$\pi_p(D) = \pi_p(D) \|id\| \geq \pi_p(D \circ id) \geq \kappa_p \|D \circ id\|_{HS} = \kappa_p \left( \sum_i |d_i|^2 \right)^{1/2}.$$

■

*Proof of Theorem 1.5.* By Lemma 2.2, it suffices to show that, for any convex sequence  $(\alpha_k)$  convergent to 0, there exists  $T \in B(X, Y)$  with the property that, for every  $m$ ,

$$c_p \alpha_{9m} \leq c_m^{(\Pi_p)}(T) \leq a_m^{(\Pi_2)}(T) \leq 3\alpha_{\lceil m/2 \rceil}.$$

Set  $n_0 = 0$ , and find a “rapidly increasing” sequence  $(n_k)$  with the property that, for any  $k \in \mathbb{N}$ ,  $n_k > 5(n_{k-1} + 1)$ , and  $\alpha_{n_k} \leq \alpha_{5(n_{k-1}+1)}/5$  (the first inequality follows from the second if  $\alpha_i > 0$  for every  $i$ ).

Fix  $c \in (1, 6/5)$ , and find, for each  $k$ ,  $n_k$ -dimensional spaces  $E_k \hookrightarrow X$  and  $F_k \hookrightarrow Y$ , whose Banach-Mazur distance to  $\ell_2^{n_k}$  is less than  $\sqrt{c}$ . As in Lemma 2.1, select the  $F_k$ ’s in such a way that there exist finite rank operators  $R_k \in B(Y)$ , such that  $\|R_k\| < 5/2$ ,  $R_k|_{F_k} = I_{F_k}$ , and  $R_k|_{F_s} = 0$  if  $k \neq s$ . Find contractions  $U_k : X \rightarrow \ell_2^{n_k}$  and  $V_k : \ell_2^{n_k} \rightarrow F_k$ , for which  $\|U_k^{-1}\|, \|V_k^{-1}\| < \sqrt{c}$ . Denote by  $id$  the formal identity map from  $\ell_2^{n_k}$  to  $\ell_\infty^{n_k}$ . Then  $id \circ U_k$  extends to a contraction  $W_k : X \rightarrow \ell_\infty^{n_k}$ .

For  $1 \leq j \leq n_k$ , let  $\beta_{jk} = \sqrt{\alpha_{j+2n_{k-1}}^2 - \alpha_{j+2n_{k-1}+1}^2}$ . As the sequence  $(\alpha_j)$  is convex,  $\beta_{1k} \geq \dots \geq \beta_{n_k k}$ . Let  $D_k = \text{diag}(\beta_{jk})_{j=1}^{n_k}$  be a diagonal map from  $\ell_\infty^{n_k}$  to  $\ell_2^{n_k}$ . Consider the map  $T = \sum_{k=1}^\infty V_k D_k W_k$ . As  $\pi_2(D_k)^2 = \sum_{j=1}^{n_k} \beta_{jk}^2 \leq \alpha_{2n_{k-1}+1}^2$ , the operator  $T$  is 2-summing. We shall show that  $T$  has the desired properties.

First estimate  $c_j^{(\Pi_p)}(T)$  from below. To this end, find  $k$  such that  $n_{k-1} < j \leq n_k$ . Suppose  $Z \hookrightarrow Y$  has codimension less than  $j$ . Then  $H = U_k(E_k \cap Z)$  is a subspace of  $\ell_2^{n_k}$

of codimension less than  $j$ . As  $5\|U_k^{-1}\|\|V_k^{-1}\|/2 < 3$ ,

$$\begin{aligned}\pi_p(T|_Z) &\geq \pi_p(T|_{E_k \cap Z}) \geq \frac{2}{5}\pi_p(R_k T|_{E_k \cap Z}) \geq \frac{2}{5}\pi_p(R_k V_k D_k|_H U_k) \\ &\geq \frac{1}{3}\pi_p(D_k|_H) = \frac{\kappa_p}{3}\|D_k|_H\|_{HS}\end{aligned}$$

(here, we view  $D_k$  as an operator on  $\ell_2^{n_k}$ , and  $\kappa_p$  is the constant from Lemma 2.3). Weyl's Minimax Principle implies

$$\|D_k|_H\|_{HS}^2 \geq \sum_{i=j}^{n_k} \beta_{ik}^2 = \alpha_{j+2n_{k-1}}^2 - \alpha_{2n_{k-1}+n_k+1}^2.$$

Let  $m_k$  be the largest  $m \leq n_k$  for which  $\alpha_{m+2n_{k-1}} \geq 1.1\alpha_{n_k+2n_{k-1}+1}$  (by construction,  $m_k > n_{k-1}$ ). Consider three cases: (i)  $n_{k-1} < j \leq m_k$ , (ii)  $m_k < j \leq n_k$  and  $m_k \geq n_k/3$ , and (iii)  $m_k < j \leq n_k$  and  $m_k < n_k/3$ .

If  $n_{k-1} < j \leq m_k$ , we obtain

$$\|D_k|_H\|_{HS} \geq \sqrt{1 - (10/11)^2} \alpha_{j+2n_{k-1}} \geq \frac{1}{3} \alpha_{j+2n_{k-1}} \geq \frac{1}{3} \alpha_{3j-2},$$

hence  $\pi_p(T|_Z) \geq \kappa_p \alpha_{j+2n_{k-1}} \geq \kappa_p \alpha_{3j-2}/9$ . As this inequality holds whenever  $\dim Y/Z < j$ , we conclude that  $c_j^{(\Pi_p)}(T) \geq \kappa_p \alpha_{3j-2}/9$ .

Now suppose  $m_k < j \leq n_k$ . As  $n_k < m_{k+1}$ , the reasoning above yields

$$(2.9) \quad c_j^{(\Pi_p)}(T) \geq c_{n_k+1}^{(\Pi_p)}(T) \geq \kappa_p \alpha_{3n_k+1}/9.$$

If  $j > m_k \geq n_k/3$ , we conclude that  $c_j^{(\Pi_p)}(T) \geq \kappa_p \alpha_{9j}/9$ .

It remains to consider the case when  $m_k \leq n_k/3$ . Then (2.7) implies

$$\frac{\alpha_{m_k+2n_{k-1}+1} - \alpha_{n_k+2n_{k-1}+1}}{n_k - m_k} \geq \frac{\alpha_{n_k+1} - \alpha_{3n_k+1}}{2n_k},$$

hence

$$\begin{aligned}\alpha_{n_k+1} - \alpha_{3n_k+1} &\leq \frac{2n_k}{n_k - m_k} (\alpha_{m_k+2n_{k-1}+1} - \alpha_{n_k+2n_{k-1}+1}) \\ &\leq \frac{2}{2/3} (1.1 - 1) \alpha_{n_k+2n_{k-1}+1} \leq 0.3 \alpha_{n_k+1},\end{aligned}$$

hence

$$(2.10) \quad \alpha_{3n_k+1} \geq 0.7 \alpha_{n_k+1}$$

Using (2.9), we obtain, for  $j > m_k$ ,

$$\begin{aligned}\kappa_p^{-1} c_j^{(\Pi_p)}(T) &\geq \kappa_p^{-1} c_{n_k+1}^{(\Pi_p)}(T) \geq \frac{1}{9} \alpha_{3n_k+1} \geq \frac{7}{90} \alpha_{n_k+1} \\ &\geq \frac{7}{90} \alpha_{n_k+2n_{k-1}+1} \geq \frac{7}{90 \cdot 1.1} \alpha_{m_k+2n_{k-1}+1} \geq \frac{7}{99} \alpha_{3j}\end{aligned}$$

(here, we use the fact that  $m_k > n_{k-1}$ ).

Next we estimate  $a_j^{(\Pi_2)}(T)$  from above. Denote by  $P_{sk}$  the projection onto the first  $s$  coordinates of  $\ell_\infty^{n_k}$ . Then

$$\pi_2(D_k(I - P_{sk}))^2 = \sum_{j=s+1}^{n_k} \beta_{jk}^2 = \alpha_{s+2n_{k-1}}^2 - \alpha_{n_k+2n_{k-1}+1}^2 \leq \alpha_{s+2n_{k-1}}^2.$$

If  $n_1 + \dots + n_{k-1} < j \leq n_k$ , then

$$u = \sum_{s < k} U_s D_s W_s + U_k D_k P_{j-(1+n_1+\dots+n_{k-1}),k} W_k.$$

has rank less than  $j$ , hence

$$\begin{aligned} a_j^{(\Pi_2)}(T) &\leq \pi_2(T - u) \leq \sum_{s > k} \|V_s\| \pi_2(D_s) \|W_s\| + \|U_k\| \pi_2(D_k(I - P_{j-(1+n_1+\dots+n_{k-1}),k})) \|W_k\| \\ &\leq \sum_{s \geq k} \alpha_{2n_s+1} + \alpha_{j+n_{k-1}-(1+n_1+\dots+n_{k-2})} \leq 3\alpha_j \end{aligned}$$

(here, we use the fact that  $n_{k-1} > 2(1 + n_1 + \dots + n_{k-2})$ , and  $\alpha_{n_s+1} \leq \alpha_{5(n_s+1)}/5$ , for each  $s$ ). If  $n_{k-1} < j \leq n_1 + \dots + n_{k-1}$ , then, by the above reasoning,

$$a_j^{(\Pi_2)}(T) \leq a_{n_{k-1}}^{(\Pi_2)}(T) \leq 3\alpha_{n_{k-1}} \leq 3\alpha_{[4j/5]},$$

since  $n_{k-1} > 4(n_1 + \dots + n_{k-1})/5$ . ■

To establish Theorems 1.6 and 1.7, we need to prove two lemmas.

**Lemma 2.4.** *Suppose  $X$  is a Banach space without non-trivial cotype, and  $E$  and  $X'$  are subspaces of  $X$  of finite dimension and codimension, respectively. Then, for every  $n \in \mathbb{N}$  and  $\varepsilon > 0$ , there exists a  $n$ -dimensional subspace  $F \hookrightarrow X'$ , such that  $d(F, \ell_\infty^n) < 1 + \varepsilon$ , and there exists a projection  $P$  from  $X$  onto  $F$ , such that  $\|P\| < 1 + \varepsilon$ , and  $P|_E = 0$ .*

**Lemma 2.5.** *Suppose a Banach space  $X$  has Property  $(\mathcal{P})_C$ , and  $E$  and  $X'$  are subspaces of  $X$  of finite dimension and codimension, respectively. Then, for every  $n \in \mathbb{N}$  and  $\varepsilon > 0$ , there exists a  $n$ -dimensional subspace  $F \hookrightarrow X'$ , such that  $d(F, \ell_2^n) \leq C$ , and there exists a projection  $P$  from  $X$  onto  $F$ , such that  $\|P\| < C^2 + \varepsilon$ , and  $P|_E = 0$ .*

To establish these two lemmas, we need a “small perturbation” result.

**Lemma 2.6.** *Suppose  $E$  and  $F$  are subspaces of a Banach space  $X$ , with  $\dim F = n < \infty$ , and  $P$  is a projection from  $X$  onto  $F$ , with  $\|P|_E\| < \varepsilon$  ( $0 < \varepsilon < 1/8$ ). Then there exists a projection  $Q$  from  $X$  onto  $F$ , such that  $Q|_E = 0$ , and  $\|P - Q\| \leq 4\|P\|n\varepsilon$ .*

*Proof.* Note first that, for any  $e \in E$  and  $f \in F$ ,

$$(2.11) \quad \|e + f\| \geq (\|e\| + \|f\|)/(4\|P\|)$$

Indeed,

$$\|P\|\|e + f\| \geq \|P(e + f)\| \geq \|f\| - \|Pe\| \geq \|f\| - \varepsilon\|e\|.$$

Moreover,

$$(1 + \|P\|)\|e + f\| \geq \|I - P\|\|e + f\| \geq \|(I - P)(e + f)\| \geq (1 - \varepsilon)\|e\|.$$

Therefore,

$$\begin{aligned} \|e + f\| &= \frac{\|P\|}{2\|P\| + 1} \|e + f\| + \frac{\|P\| + 1}{2\|P\| + 1} \|e + f\| \\ &\geq \frac{1}{2\|P\| + 1} (\|f\| - \varepsilon\|e\| + (1 - \varepsilon)\|e\|) = \frac{\|f\| + (1 - 2\varepsilon)\|e\|}{2\|P\| + 1} \geq \frac{3}{4} \cdot \frac{\|f\| + \|e\|}{3\|P\|}, \end{aligned}$$

yielding (2.11).

Fix an Auerbach basis  $(f_i)_{i=1}^n$  in  $F$ . Then there exist norm 1 elements  $f_i^* \in F^*$  satisfying  $\langle f_i^*, f_j \rangle = \delta_{ij}$  (Kronecker's delta). Let  $x_i^* = P^* f_i^*$ . Then  $\|x_i^*\| \leq \|P\|$  ( $1 \leq i \leq n$ ), and, for every  $x \in X$ ,  $Px = \sum_{i=1}^n \langle f_i^*, Px \rangle f_i = \sum_{i=1}^n \langle x_i^*, x \rangle f_i$ . Therefore,  $\|x_i^*|_E\| \leq \|P|_E\| < \varepsilon$ . Define  $y_i^* \in (E + F)^*$  by setting  $x_i^*|_E = y_i^*|_E$ , and  $y_i^*|_F = 0$ . By (2.11),  $\|y_i^*\| \leq 4\|P\|\varepsilon$ . By Hahn-Banach Theorem, there exist  $z_i^* \in X^*$  ( $1 \leq i \leq n$ ) such that  $z_i^*|_E = x_i^*|_E$ ,  $z_i^*|_F = 0$ , and  $\|z_i^*\| \leq 4\|P\|\varepsilon$ . Then the projection  $Q$ , defined by  $Qx = \sum_{i=1}^n \langle x_i^* - z_i^*, x \rangle f_i$ , has the desired properties.  $\blacksquare$

*Proof of Lemma 2.4.* Fix  $n \in \mathbb{N}$  and  $\delta \in (0, 1/8)$ . By compactness, there exists  $M \in \mathbb{N}$  such that, for every collection  $(z_s)_{s=1}^M$  in  $B(E^*)$ , there exist  $n$  pairs

$$(p_i, q_i) \in \{1, \dots, M\}^2 \setminus \{(1, 1), \dots, (M, M)\} \quad (1 \leq i \leq n),$$

such that  $\{p_i, q_i\} \cap \{p_j, q_j\} = \emptyset$  unless  $i = j$ , and  $\|z_{p_i} - z_{q_i}\| < \delta/n$  for every  $i$ .

By Krivine-Maurey-Pisier Theorem (see e.g. [16]), for every  $\delta > 0$  there exists  $G \hookrightarrow X'$  with  $d(G, \ell_\infty^M) < 1 + \delta$ . Find a contraction  $U : G \rightarrow \ell_\infty^M$  such that  $\|U^{-1}\| < 1 + \delta$ , and extend it to a contraction  $\tilde{U} : X \rightarrow \ell_\infty^M$ . There exist  $(x_i^*)_{i=1}^M$  in the unit ball of  $X^*$  such that  $\tilde{U}x = \sum_{i=1}^M \langle x_i^*, x \rangle \sigma_i$ , where  $(\sigma_i)$  is the canonical basis on  $\ell_\infty^M$  (hence,  $\langle x_i^*, U^{-1}\sigma_j \rangle$  equals 1 if  $i = j$ , 0 otherwise).

By our choice of  $M$ , there exist disjoint pairs  $(p_i, q_i)$  ( $1 \leq i \leq n$ ) such that, for each  $i$ ,  $\|(x_{p_i}^* - x_{q_i}^*)|_E\| < \delta/n$ . Let  $\tilde{F} = \text{span}[\sigma_{p_i} - \sigma_{q_i} : 1 \leq i \leq n] \hookrightarrow \ell_\infty^M$ , and  $F = U^{-1}(\tilde{F})$ . Then  $\tilde{F}$  is isometric to  $\ell_\infty^n$ , and  $d(F, \ell_\infty^n) < 1 + \delta$ . Furthermore,  $\tilde{F}$  is the range of the contractive projection  $Q$ , defined by setting

$$Q\sigma_j = \begin{cases} 0 & j \notin \cup_i \{p_i, q_i\} \\ (\sigma_{p_i} - \sigma_{q_i})/2 & j = p_i \\ -(\sigma_{p_i} - \sigma_{q_i})/2 & j = q_i \end{cases}.$$

Then  $P = U^{-1}Q\tilde{U}$  is a projection onto  $F$ , with  $\|P\| < 1 + \delta$ . Moreover,

$$Px = \frac{1}{2} \sum_{i=1}^n \langle x_{p_i}^* - x_{q_i}^*, x \rangle U^{-1}(\sigma_{p_i} - \sigma_{q_i})$$

for  $x \in X$ , hence  $\|P|_E\| < \delta$ . As  $\delta > 0$  can be chosen to be arbitrarily small, an application of Lemma 2.6 completes the proof.  $\blacksquare$

*Proof of Lemma 2.5.* Fix  $n \in \mathbb{N}$  and  $\delta \in (0, 1/8)$ . Select a  $\delta C^{-2}$ -net  $(e_i)_{i=1}^M$  in  $B(E)$ . Pick  $m > MC^4/\delta^2$ . Find  $G \hookrightarrow X'$ , which is  $C$ -isomorphic to  $\ell_2^{mn}$ , and  $C$ -complemented in  $X$ . Consider a contraction  $U : G \rightarrow \ell_2^{mn}$  such that  $\|U^{-1}\| \leq C$ , and a projection  $P$  from  $X$  onto  $G$ , with  $\|P\| \leq C$ . Denote by  $(\sigma_j)_{j=1}^{mn}$  the canonical basis for  $\ell_2^{mn}$ , and let  $Q_k$  ( $1 \leq k \leq m$ ) be the orthogonal projection from  $\ell_2^{mn}$  onto  $\text{span}[\sigma_i : (k-1)n+1 \leq i \leq kn]$ . Then, for every  $k$ ,  $P_k = U^{-1}Q_k U P$  is a projection of norm not exceeding  $C^2$ , whose range is  $C$ -isomorphic to  $\ell_2^n$ . We claim that there exists  $k$  such that  $\|P_k e\| < 2\delta$  for any  $e \in B(E)$ . Once the existence of such  $k$  is established, we can complete the proof by applying Lemma 2.6 to  $P_k$  and  $F_k = \text{ran } P_k = U^{-1}\text{span}[\sigma_i : (k-1)n+1 \leq i \leq kn]$ .

Note that, if  $x_k \in \text{ran } P_k$  for  $1 \leq k \leq m$ , then

$$\left\| \sum_{k=1}^m x_k \right\| \geq \left\| \sum_k Ux_k \right\| = \left( \sum_{k=1}^m \|Ux_k\|^2 \right)^{1/2} \geq C^{-1} \left( \sum_{k=1}^m \|x_k\|^2 \right)^{1/2}.$$

Thus, for any  $x \in B(X)$ ,  $C^4 \geq C^2 \|Px\|^2 \geq \sum_{k=1}^m \|P_k x\|^2$ , hence  $\|P_k x\| \geq \delta$  for at most  $C^4/\delta^2$  values of  $k$ . As  $m > MC^4/\delta^2$ , there exists  $k$  such that  $\|P_k e_i\| < \delta$  for every  $i \in \{1, \dots, M\}$ . For every  $e \in B(E)$ , find  $i$  such that  $\|e - e_i\| < \delta/(4C^2)$ . Then

$$\|P_k e\| \leq \|P_k e_i\| + \|P_k\| \|e - e_i\| < \delta + C^2 \cdot \frac{\delta}{C^2} < 2\delta,$$

as desired. ■

*Proof of Theorem 1.6.* By Lemma 2.2, it suffices to show that, for any convex sequence  $\alpha_i \searrow 0$ , there exists  $T \in B(X, Y)$  satisfying

$$4\alpha_{\lceil 4m/5 \rceil} \geq a_m^{(A)}(T) \geq c_m(T) \geq \frac{49}{1100} \alpha_{9m}.$$

Find a sequence  $0 = n_0 < n_1 < n_2 < \dots$  such that, for each  $k$ ,  $\alpha_{n_k} \leq \alpha_{5(n_{k-1}+1)}/5$ , and  $n_k > 5(n_{k-1} + 1)$ . Find sequences of subspaces  $E_k \hookrightarrow X$  and  $F_k \hookrightarrow Y$  in such a way that:

- (1) There exist contractions  $U_k : E_k \rightarrow \ell_\infty^{n_k}$  and  $V_k : \ell_1^{n_k} \rightarrow F_k$  such that their inverses have norms less than  $2^{1/4}$ .
- (2) For each  $k$ , there exists a projection  $P_k$  onto  $E_k$  such that  $\|P_k\| < 2^{1/4}$ , and  $P_k|_{E_j} = 0$  for  $j \neq k$  (in other words,  $P_j P_k = 0$  if  $j \neq k$ ).

The existence of  $(F_k)$  follows from the fact that  $Y$  has no non-trivial type [16]. Select  $(E_k)$  inductively. Select  $E_1$  to be arbitrary, subject to the estimate on  $d(E_1, \ell_\infty^{n_1})$ . Now suppose  $E_1, \dots, E_{k-1}, P_1, \dots, P_{k-1}$  have already been defined. By Lemma 2.4, there exists  $E_k \hookrightarrow \cap_{j=1}^{k-1} \ker P_j$ , and a projection  $P_k$  onto it, such that  $d(E_k, \ell_\infty^{n_k}) < 2^{1/4}$ ,  $\|P_k\| < 2^{1/4}$ , and  $P_k|_{E_j} = 0$  for any  $j < k$ .

For  $1 \leq i \leq n_k$ , set  $\beta_{ik} = \alpha_{i+2n_{k-1}} - \alpha_{i+2n_{k-1}+1}$ . By the convexity of  $(\alpha_i)$ ,  $\beta_{1k} \geq \beta_{2k} \geq \dots \geq \beta_{n_k k}$ . Let  $D_k = \text{diag}(\beta_{ik})$  be the diagonal map from  $\ell_\infty^{n_k}$  to  $\ell_1^{n_k}$ , and set  $S_k = V_k D_k U_k$  (we can view  $S_k$  as a map into  $Y$ ). We claim that the operator  $T = \sum_j S_j P_j$  has the desired properties. To this end, recall that (see e.g. [22, Section 11.11]), for a diagonal operator  $D = \text{diag}(d_i) \in B(\ell_\infty^n, \ell_1^n)$ ,  $a_m(D) = c_m(D) = \sum_{i=m}^n d_i$  (here, we are assuming that  $m \leq n$ , and  $d_1 \geq d_2 \geq \dots \geq d_n$ ). Furthermore, for any ideal  $\mathcal{A}$ ,  $\|D\|_{\mathcal{A}} \leq \sum_i d_i$  (to see this, represent  $D$  as a sum of rank 1 diagonal operators), hence  $c_m(D) = a_m^{(A)}(D) = \sum_{i=m}^n d_i$ .

To estimate  $c_m(T)$  from below, find  $k$  such that  $n_{k-1} < m \leq n_k$ . By the injectivity of  $\mathcal{A}$ ,

$$c_m(T) \geq c_m(T|_{E_k}) = c_m(S_k) \geq 2^{-1/2} c_m^{(A)}(D_k) \geq 2^{-1/2} c_m(D_k).$$

As noted above,  $c_m(D_k) = \sum_{i=m}^{n_k} \beta_{ik} = \alpha_{m+2n_{k-1}} - \alpha_{n_k+2n_{k-1}+1}$ . As in the proof of Theorem 1.5, let  $m_k$  be the largest number  $m \leq n_k$  for which  $\alpha_{m+2n_{k-1}} \geq 1.1\alpha_{n_k+2n_{k-1}+1}$ . If  $m \leq m_k$ , then  $c_m(D_k) \geq 0.1\alpha_{m+2n_{k-1}} \geq 0.1\alpha_{3m}$ , hence  $c_m(T) \geq 0.07\alpha_{3m}$ . For  $m > m_k$ , recall that  $n_k + 1 \leq m_{k+1}$ , hence  $c_m(T) \geq c_{n_k+1}(T) \geq 0.07\alpha_{3n_k+1}$ . If  $m > n_k/3$ , this yields

$c_m(T) \geq 0.07\alpha_{9m}$ . If  $m_k < m \leq n_k/3$ , (2.10) implies  $\alpha_{3n_k+1} \geq 0.7\alpha_{n_k+1}$ . Therefore,

$$c_m(T) \geq c_{n_k+1}(T) \geq 0.07\alpha_{3n_k+1} \geq \frac{7^2}{1000}\alpha_{n_k+2n_{k-1}+1} \geq \frac{49}{1100}\alpha_{m_k+2n_{k-1}+1} \geq \frac{49}{1100}\alpha_{3m}.$$

Next estimate  $a_m^{(A)}(T)$  from above. Suppose  $n_1 + \dots + n_{k-1} < m \leq n_k$ . Then

$$a_m^{(A)}(T) \leq a_{m-(n_1+\dots+n_{k-1})}^{(A)}(S_k P_k) + \sum_{j=k}^{\infty} \|S_{j+1} P_{j+1}\|_{\mathcal{A}} \leq 2^{3/4} (a_{\ell}^{(A)}(D_k) + \sum_{j=k}^{\infty} \|D_{j+1}\|_{\mathcal{A}}),$$

where  $\ell = m - (n_1 + \dots + n_{k-1})$ . But  $\|D_{j+1}\|_{\mathcal{A}} \leq \alpha_{n_j+2n_{j-1}+1} \leq \alpha_{n_j+1}$ . Moreover,  $\alpha_{n_{s+1}+1} \leq \alpha_{n_s+1}/5$  for any  $s$ , hence  $\sum_{j=k+1}^{\infty} \|D_j\|_{\mathcal{A}} \leq 5\alpha_{n_k+1}/4 \leq 5\alpha_m/4$ . As noted previously,

$$a_{\ell}^{(A)}(D_k) = \sum_{i=\ell}^{n_k} \beta_{in_k} \leq \alpha_{m+n_{k-1}-n_{k-2}-\dots-n_1} \leq \alpha_m.$$

Therefore,  $a_m^{(A)}(T) \leq 2^{3/4}(\alpha_m + 5\alpha_m/4) \leq 4\alpha_m$ .

Now suppose  $n_{k-1} < m \leq n_1 + \dots + n_{k-1}$ . Then  $a_m^{(A)}(T) \leq a_{n_{k-1}+1}^{(A)}(T) \leq 4\alpha_{n_{k-1}+1}$ . As  $m \leq 5n_{k-1}/4$ , we conclude that  $a_m^{(A)}(T) \leq 4\alpha_{\lceil 4m/5 \rceil}$ . ■

*Proof of Theorem 1.7.* The proof is very similar to the proof of Theorem 1.6. Suppose  $X$  has Property  $(\mathcal{P})_C$ . By Lemma 2.2, it suffices to show that, for any convex sequence  $\alpha_i \searrow 0$ , there exists an operator  $T \in B(X, Y)$  such that

$$\frac{1}{10C^2}\alpha_{9m} \leq c_m^{(\Pi_{tr})}(T) \leq a_m^{(\Pi_{tr})}(T) \leq 4\alpha_{\lceil 4m/5 \rceil}$$

To this end, pick  $C_1 \in (C^2, 70C^2/66)$ . Find a sequence  $0 = n_0 < n_1 < n_2 < \dots$  such that, for each  $k$ ,  $\alpha_{n_k} \leq \alpha_{5(n_{k-1}+1)}/5$ , and  $n_k > 5(n_{k-1} + 1)$ . Find sequences of subspaces  $E_k \hookrightarrow X$  and  $F_k \hookrightarrow Y$  in such a way that:

- (1) There exist contractions  $U_k : E_k \rightarrow \ell_2^{n_k}$  and  $V_k : \ell_2^{n_k} \rightarrow F_k$ , such that  $\|U_k^{-1}\| \leq C$ , and  $\|V_k^{-1}\| < 2$ .
- (2) For each  $k$ , there exists a projection  $P_k$  onto  $E_k$  such that  $\|P_k\| \leq C_1$ , and  $P_k P_j = 0$  for  $k \neq j$ .

The existence of  $(F_k)$  follows from Dvoretzky's theorem. Select  $(E_k)$  inductively. Pick an arbitrary  $E_1$ , satisfying  $d(E_1, \ell_2^{n_1}) \leq C_1$ . Now suppose  $E_1, \dots, E_{k-1}, P_1, \dots, P_{k-1}$  have already been defined. By Lemma 2.5, there exists  $E_k \hookrightarrow \cap_{j=1}^{k-1} \ker P_j$ , and a projection  $P_k$  onto it, such that  $d(E_k, \ell_2^{n_k}) \leq C_1$ ,  $\|P_k\| \leq C_1$ , and  $P_k|_{E_j} = 0$  for any  $j < k$ .

For  $1 \leq i \leq n_k$ , let  $\beta_{ik} = (\alpha_{i+2n_{k-1}}^q - \alpha_{n_k+2n_{k-1}+1}^q)^{1/q}$ , where  $1/q = 1/2 - 1/r + 1/t$ . By convexity,  $\beta_{1k} \geq \beta_{2k} \geq \dots \geq \beta_{n_k k}$ . Let  $D_k = \text{diag}(\beta_{ik})$  be the diagonal map on  $\ell_2^{n_k}$ , and set  $S_k = V_k D_k U_k$  (we can view  $S_k$  as a map into  $Y$ ). We claim that the operator  $T = \sum_j S_j P_j$  has the desired properties.

We rely on a result of Mitiagin [26, Theorem 11.9]: for an operator  $u$  on a Hilbert space,  $\|u\|_q \leq \pi_{t,r}(u) \leq \mathfrak{a}^{-1}\|u\|_q$ , where  $\mathfrak{a} = \sqrt{2/\pi}$  is the first absolute Gaussian moment.



First estimate  $c_m^{(\Pi_{tr})}(T)$  from below. For a fixed  $m$ , find  $k$  such that  $n_{k-1} < m \leq n_k$ . By the injectivity of  $\Pi_{t,r}$ ,  $c_m^{(\Pi_{tr})}(T) \geq c_m^{(\Pi_{tr})}(T|_{E_k}) = c_m^{(\Pi_{tr})}(S_k) \geq c_m^{(\Pi_{tr})}(D_k)/(2C_1)$ , and

$$c_m^{(\Pi_{tr})}(D_k) = \inf_{\text{codim } H < m} \pi_{tr}(D_k|_H) \geq \inf_{\text{codim } H < m} \|D_k|_H\|_q$$

By [6],

$$\inf_{\text{codim } H < m} \|D_k|_H\|_q^q = \|\text{diag}(\beta_{ik})_{i=m}^{n_k}\|_q^q = \alpha_{m+2n_{k-1}}^q - \alpha_{n_k+2n_{k-1}+1}^q$$

Let  $m_k$  be the largest value of  $m \leq n_k$  for which  $\alpha_{m+2n_{k-1}} \geq 1.1\alpha_{n_k+2n_{k-1}+1}$ . Emulate the proof of Theorems 1.5. More precisely: if  $n_{k-1} < m \leq m_k$ , we have

$$c_m^{(\Pi_{tr})}(D_k) \geq (1 - (10/11)^q)^{1/q} \alpha_{m+2n_{k-1}+1} \geq \alpha_{3m}/3,$$

hence  $c_m^{(\Pi_{tr})}(T) \geq \alpha_{3m}/(6C_1)$ . If  $m > n_k/3$ , we conclude that

$$c_m^{(\Pi_{tr})}(T) \geq c_{n_k+1}^{(\Pi_{tr})}(T) \geq \alpha_{3n_k+1}/(6C_1) \geq \alpha_{9m}/(6C_1).$$

If  $m_k < m \leq n_k/3$ , (2.10) yields  $\alpha_{3n_k+1} > 0.7\alpha_{n_k+1}$ , and therefore,

$$6C_1 c_m^{(\Pi_{tr})}(T) \geq 6C_1 c_{n_k+1}^{(\Pi_{tr})}(T) \geq \alpha_{3n_k+1} \geq \frac{7}{10} \alpha_{n_k+1} \geq \frac{7}{10 \cdot 1.1} \alpha_{m_k+2n_{k-1}+1} \geq \frac{7}{11} \alpha_{3m}.$$

Next estimate  $a_m^{(\Pi_{tr})}(T)$  from above. If  $n_1 + \dots + n_{k-1} < m \leq n_k$ , we obtain

$$a_m^{(\Pi_{tr})}(T) \leq a_{m-(n_1+\dots+n_{k-1})}^{(\Pi_{tr})}(S_k P_k) + \sum_{j=k+1}^{\infty} \pi_{tr}(S_j P_j) \leq C_1 (a_\ell^{(\Pi_{tr})}(D_k) + \sum_{j=k+1}^{\infty} \pi_{tr}(D_j)),$$

where  $\ell = m - (n_1 + \dots + n_{k-1})$ . But, for  $j \geq k$ ,

$$\alpha \pi_{tr}(D_{j+1}) \leq \|D_{j+1}\|_q \leq \alpha_{n_j+2n_{j-1}+1} \leq \alpha_{n_j+1} \leq 5^{k-j} \alpha_m.$$

Furthermore,

$$\inf_{\text{rank } u < \ell} \|D_k - u\|_q^q = \|\text{diag}_{i=\ell}^{n_k}(\beta_{ik})\|_q^q = \sum_{i=\ell}^{n_k} \beta_{ik}^q \leq \alpha_{\ell+2n_k}^q \leq \alpha_m^q,$$

hence  $a_m^{(\Pi_{tr})}(D_k) \leq \alpha^{-1} \alpha_m$ . Therefore,  $a_m^{(\Pi_{tr})}(T) \leq \alpha^{-1} \alpha_m (1 + \sum_{s=0}^{\infty} 5^{-s}) \leq 4\alpha_m$ .

For  $n_{k-1} < m \leq n_1 + \dots + n_{k-1}$ , we have  $a_m^{(\Pi_{tr})}(T) \leq a_{n_{k-1}}^{(\Pi_{tr})}(T) \leq 4\alpha_{n_{k-1}} \leq 4\alpha_{[4m/5]}$ . ■

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